October 1997
THU-97/27
hep-th/9710057

Matrix Theory on Non-Orientable Surfaces

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Abstract

We construct the Matrix theory descriptions of M-theory on the Möbius strip and the Klein bottle. In a limit, these provide the matrix string theories for the CHL string and an orbifold of type IIA string theory.

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1. Introduction

Different string theories have been claimed to be related via one unifying theory, M-theory. This theory is supposed to reproduce the various weak coupling string theories in certain limits of its moduli space, and to give the correct interpolation in between, at finite values of the coupling (see e.g. [1]). The proposal of matrix theory as a definition of M-theory in the infinite momentum frame [2, 3] has allowed many of the claims to be verified. The matrix formulations of type II and heterotic strings in ten dimensions were constructed in [4], and compactifications to lower dimensions on tori and orbifolds were investigated as well (see [3] and references therein).

Here we would like to concentrate on compactifications of M-theory to nine dimensions on non-orientable surfaces: the Möbius strip and the Klein bottle. Type II compactifications on these manifolds, in the form of certain orientifold models, were considered in [5, 6]. These authors also argued which nine dimensional string theories would appear as limits of M-theory on these surfaces: the Möbius strip yields the nine-dimensional CHL string [7], with gauge group E_8 (arising as the twisted sector living on the (single) boundary of the strip, following [8]), and the Klein bottle represents, in a suitable limit, a type IIA string in nine dimensions modded out by half a shift over the circle accompanied by the operation $(-1)^{F_L}$.

We use the orientifold models to obtain the matrix description of these theories. The models are constructed as torus compactifications of matrix theory, modded out by an appropriate symmetry group consisting of a reflection and shifts over half a period of the circles. We first consider the Möbius strip. We construct the $2+1$ dimensional gauge theory describing the dynamics of zero-branes on the Möbius strip. The base manifold of the gauge theory is itself again a Möbius strip. The construction is similar to that of heterotic matrix theory [9]. We show how in the limit of weak string coupling we indeed recover a string theory with chiral fermions producing one E_8 gauge group, the CHL string.

Then we turn to the Klein bottle compactification. In this case the T-duality that we have to perform to construct the $2+1$ dimensional theory describing the dynamics is less straightforward, we have to use the original construction of [10, 11] to find the gauge theory. The base manifold of the gauge theory is *not* a Klein bottle. Rather, the geometric structure of the Klein bottle is reflected in the structure of the gauge fields; different modes of the fields turn out to satisfy different conditions. Finally we again find the type IIA string theory, modulo the required symmetry, as a limit of this three-dimensional model.

After the preprint of this paper was made public on the archives, [12] ap-

peared, which also studies the matrix descriptions of M-theory on the Möbius strip and the Klein bottle. These authors find similar results to ours in the case of the Möbius strip. For the Klein bottle, however, they argue that the gauge theory base space is a Klein bottle, and not a cylinder as we found. A possible resolution to this contradiction was suggested in [13]. There it was found, using non-commutative geometry techniques, that the topology of the base space depends on the value of the background anti-symmetric tensor field B through the original Klein bottle. In the absence of this background the result exactly coincides with our conclusions, i.e. the gauge theory base space is a cylinder with the same field content as presented in this paper. If a half-integral B-field Wilson surface is switched on, however, the result is a Klein bottle. Presumably in the work of [12] there is such a B-field, although it is not obvious to us where it enters their argument.

2. M-theory on a Möbius strip

In constructing the M(atrix)-theory compactified on a Möbius strip we will start from a related type IA theory studied by [5, 6], and consider the dynamics of D0-branes in that model.

The IA theory in question is given by the type IIA theory compactified on a two torus, with radii $R_{1,2}$, divided out by the symmetries $\Omega\mathcal{I}_1$ and $\mathcal{S}_1\mathcal{S}_2$. Here Ω is the world sheet orientation reversal, $\mathcal{I}_1$ inverts the first coordinate, making the first circle into a line segment, and the $\mathcal{S}_i$ are shifts in the compact directions by half a period:

$$X_1 \to X_1 + \pi R_1, \quad X_2 \to X_2 + \pi R_2.$$

In the M-theory point of view $\Omega\mathcal{I}_1$ will become the inversion of the eleventh direction, and the resulting compactification manifold is the Möbius strip.

M-theory on a circle divided by the inversion of the circle was demonstrated to be equivalent to heterotic $E_8 \times E_8$ string theory [8]. One E_8 factor lives on each ten-dimensional boundary. Weak coupling corresponds to shrinking the compact direction. In the present case we divide out one more symmetry, which exchanges the two boundaries ($\mathcal{S}_1$) and rotates by half a period in an extra compact direction ($\mathcal{S}_2$). In the heterotic string, this is exactly the operation producing a nine-dimensional CHL string [7], so we expect our matrix model to produce matrix CHL string theory in the limit of vanishing R_1.

We will first review Dabholkar and Park's analysis of the orientifold model underlying the matrix description. On closed string states with momentum numbers n_1, n_2, the action of $\mathcal{S}_1\mathcal{S}_2$ is simply $(-1)^{n_1+n_2}$. The untwisted closed string

spectrum is therefore that of the usual type IA theory in eight dimensions, except that those states with odd n_1+n_2 are projected out (note that this does not affect the massless spectrum). In addition there are (massive) twisted states, having half integer winding numbers.

The open string spectrum is obtained by calculating the Klein bottle contribution to the RR-tadpoles. The amplitude with $\Omega\mathcal{I}_1$ in the trace gives the usual 32 D8-branes, aligned along the 2-direction. Their X_1 coordinates have to be compatible with both $\mathcal{I}_1$ and $\mathcal{S}_1$; the maximal symmetry is $SO(16)$, obtained when sixteen D8-branes lie on the orientifold plane $X_1 = 0$ (and the other sixteen, by $\mathcal{S}_1$ symmetry, on $X_1 = \pi R_1$). At strong coupling the symmetry is expected to become E_8, similarly as in the regular IA theory. The $\Omega\mathcal{I}_1\mathcal{S}_1\mathcal{S}_2$ in the trace gives a contribution that vanishes in the long tube limit. Finally, the twisted channels vanish altogether, since they necessarily have non-vanishing winding number in the 2-direction, incompatible with Ω in the trace. The resulting massless spectrum is the same as that of the CHL model in eight dimensions, and the two models were in fact claimed to be dual by [5, 6].

We will now analyse the dynamics of D0-branes in this background, and study the M(atrix)-theory limit in which the model is assumed to be lifted to eleven dimensions. As is well known the full dynamics on the compact space is described by a gauge theory in $2+1$ dimensions, which is obtained by T-dualising the D0-branes [10]. Since the T-duals of the shift symmetries are not obvious to us, we find it convenient to describe the system slightly differently. Make a change of coordinates to

$$X_\pm = \frac{X_2}{R_2} \pm \frac{X_1}{R_1}. \tag{2.1}$$

If we include into the orbifold group the translations that make the $\mathbf{R}^2$ into a torus, the group is generated by

$$\mathcal{S}_1\mathcal{S}_2, \mathcal{S}_1^{-1}\mathcal{S}_2, \text{ and } \Omega\mathcal{I}_1.$$

The first two elements shift X_+ resp. X_- by one unit, creating an ordinary torus out of the X_+X_--plane. The effect of $\mathcal{I}_1$ is to exchange the two coordinates:

$$\mathcal{I}_1 \begin{pmatrix} X_+ \\ X_- \end{pmatrix} = \begin{pmatrix} X_- \\ X_+ \end{pmatrix},$$

and Ω again exchanges left and right movers. In terms of these coordinates we therefore have a type IIA theory on a torus, divided out by one (unconventional) symmetry. The operation is drawn in figure 1; taking the shaded region as fundamental domain, instead of one of the triangles, one easily verifies that this indeed represents a Möbius strip.

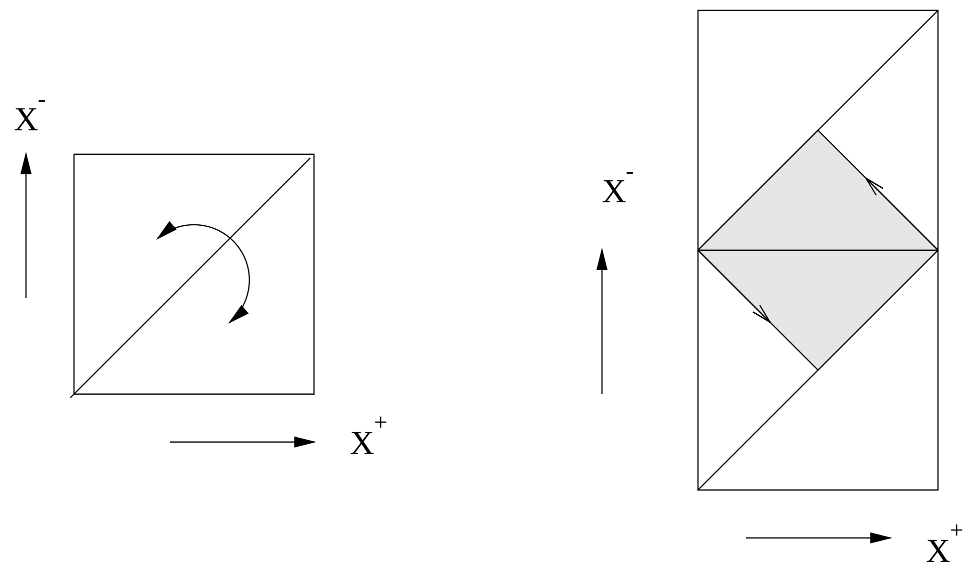

Figure 1: *On the left, the torus is drawn, with the symmetry to be divided out. The result is a Möbius strip. This can be most easily seen in the right-hand figure, where instead of the triangle we take the shaded region as fundamental domain; the arrows indicate the identification of the two sides*

The metric on the torus is, in terms of the original radii $R_{1,2}$,

$$G = \frac{1}{4} \begin{pmatrix} R_2^2 + R_1^2 & R_2^2 - R_1^2 \\ R_2^2 - R_1^2 & R_2^2 + R_1^2 \end{pmatrix}. \tag{2.2}$$

In particular, when the two radii of the original torus are equal (R), the new radii are $\tilde{R} = \frac{R}{\sqrt{2}}$.

The orientifold plane is now really only one plane, situated along the diagonal $X_+ = X_-$. This diagonal is the boundary of the Möbius strip . We again compute the open string spectrum to verify that this is indeed the correct model. In the Klein bottle calculation, the oscillator contributions are the same as before; furthermore we have a momentum sum with $p_+ = p_-$, and a winding mode sum with $w_+ = -w_-$. There are no twisted sectors, since the orientifold group has only the one element $\Omega\mathcal{I}_1$. The resulting Klein bottle amplitude is

$$\mathcal{A}_{KB} = -\frac{8V_8}{(8\pi^2\alpha')^4} \int dl (16)^2. \tag{2.3}$$

To find the normalisation we also compute the cylinder diagram (for convenience we take the case where $R_1 = R_2$). Here we have diagonal momenta (along the eight-branes) $p = \frac{n}{\sqrt{2}\tilde{R}}$, and winding (transverse to the eight-branes) over a

multiple of $\frac{\tilde{R}}{\sqrt{2}}$. The amplitude is

$$\mathcal{A}_{Cyl} = -\frac{8V_8}{(8\pi^2\alpha')^4} \int dl (\mathrm{Tr}\gamma_1)^2. \tag{2.4}$$

We clearly need only sixteen D8-branes, distributed symmetrically around the orientifold plane. The factor of two difference with the usual result comes from the difference between winding and momentum sums in the Klein bottle and cylinder. In this representation we therefore obtain a somewhat simpler picture, where again the maximal symmetry is $SO(16)$.

Now we go to the M(atrix)-theory description. We have to consider the quantum mechanics of N D0-branes, with masses $\frac{1}{R_{11}}$, on the torus modded out by the orientifold group. In the limit where $N \to \infty$ this will describe M-theory on the Möbius strip in the infinite momentum frame.

To construct the theory we first take the model describing zero-branes of type IIA on a torus, and then divide out the symmetry. Until further notice we take $R_1 = R_2 = R$. The dynamics is described by a $2+1$ $U(N)$ SYM theory with sixteen supersymmetries, whose action is the dimensional reduction from the ten-dimensional $\mathcal{N} = 1$ gauge theory:

$$S_{IIA} = -\frac{1}{4g_{YM}^2} \int \mathrm{Tr} \left(F_{\mu\nu}^2 + 2g_{YM}^2 D_\mu X_i^2 - g_{YM}^4 [X_i, X_j]^2 + \text{ fermions} \right). \tag{2.5}$$

The theory contains seven scalars and eight three-dimensional fermions, all in the adjoint representation. The two-dimensional space is the T-dual of the torus the zero-branes moved on; its radii are

$$r = \frac{\sqrt{2}\ell_{11}^3}{R_{11} R},$$

(ℓ_{11} is the eleven-dimensional Planck length) and the coupling is

$$g_{YM}^2 = \frac{2R_{11}}{R^2}.$$

We now have to factor out the required symmetry $\Omega\mathcal{I}_1$. Under T-duality of the X_+ and X_- coordinates this symmetry is converted into $-\Omega\mathcal{I}_1$. Its fixed planes are therefore the lines $X_+ = -X_-$, perpendicular to the eight-branes before T-dualising, as expected. The action on the fields is as follows:

$$\begin{array}{rcl} A_0(x_+, x_-) & \to & -A_0^T(-x_-, -x_+) \\ A_+(x_+, x_-) & \to & A_-^T(-x_-, -x_+) \\ A_-(x_+, x_-) & \to & A_+^T(-x_-, -x_+) \\ X_i(x_+, x_-) & \to & X_i^T(-x_-, -x_+) \\ \psi_I(x_+, x_-) & \to & \frac{1}{2}\sqrt{2}\gamma_+(1+\gamma_0)\psi_I^T(-x_-, -x_+). \end{array} \tag{2.6}$$

The transformation rule of the fermions is motivated by the fact that the action of $-\mathcal{I}_1$ can be obtained by first rotating the X_+X_- plane over $-\pi/2$ and then reflecting in the line $X_- = 0$. Choosing three-dimensional gamma-matrices

$$\gamma_0 = i\sigma_2, \quad \gamma_+ = \sigma_1, \quad \gamma_- = \sigma_3,$$

we find the rotation is represented by

$$\exp i\frac{\pi}{2}(\frac{i}{2}\gamma_+\gamma_-) = \begin{pmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{pmatrix} = \frac{1}{2}\sqrt{2}(1 + i\sigma_2),$$

while the reflection is implemented by multiplication by γ_+.

These transformations relate the fields on both sides of the fixed plane. (Note that these sides are actually connected). In particular they impose conditions on the fields living on the fixed plane, breaking the $U(N)$ symmetry to a subgroup. Effectively the gauge theory itself lives on a Möbius strip, with specific conditions on the fields on the boundary.

Let us then determine the two-dimensional spectrum living on the fixed line, obtained by restricting the three-dimensional fields to the boundary. First of all we have a two-dimensional vector, consisting of the three-dimensional vectors tangent to the line $X_+ = -X_-$: A_0 and $\frac{1}{2}\sqrt{2}(A_+ - A_-)$. From the transformation rules (2.6) we see that these have to be antisymmetric, so that the gauge group in two dimensions is broken to $O(N)$. The remaining bosonic fields, $\frac{1}{2}\sqrt{2}(A_+ + A_-)$ and X_i, are in the symmetric representation of this group. The three-dimensional fermions are split up in two sets of two-dimensional fermions of definite chirality. The chirality operator on the fixed line is

$$\gamma_3 = \frac{1}{2}\sqrt{2}\gamma_0(\gamma_+ - \gamma_-) = -\frac{1}{2}\sqrt{2}\gamma_+(1 + \gamma_0),$$

so that the spinors of negative chirality are in the symmetric, and those of positive chirality in the adjoint representation.

As in similar models where symmetries with fixed points are divided out, we expect extra twisted matter on the fixed line. In the type IA description these extra states arise from the quantisation of strings connecting the D2-brane to the D8-branes; each two-eight string gives rise to one massless fermion, in the fundamental representations of the D2 and D8 gauge groups. We expect therefore sixteen Majorana-Weyl spinors in the fundamental of $SO(N)$.

The M(atrix)-theory motivation for the extra matter is that the two-dimensional theory as it stands is anomalous, due to the different representations of the left- and right-handed spinors. In fact the anomaly can be cancelled by adding 32 positive chirality fermions in the fundamental representation.

There seems to be a discrepancy between the two ways of counting twisted fermions. This can be resolved by making the anomaly argument a bit more precise. The gauge anomaly is an anomaly of the three-dimensional gauge theory, supported at the boundary. To calculate the actual coefficient of the anomaly, following [8], we perform a gauge transformation whose gauge parameter Λ is constant along the direction perpendicular to the fixed plane:

$$(\partial_+ + \partial_-)\Lambda = 0; \tag{2.7}$$

then we will find the two-dimensional anomaly. But now note that (2.7) implies that Λ can be independently defined only along half of the fixed line. The line $X_+ + X_- =$ constant intersects the fixed line (or its copies under translation) in two different points. We conclude that, by symmetry, half of the anomaly is supported on the lower component of the boundary, the other half on the upper component. Therefore, to cancel the anomaly we indeed only have to add sixteen chiral fermions in the fundamental representation.

We now wish to go to the limit in moduli space where the CHL string is weakly coupled, and find the underlying matrix string theory [4]. The coupling constant of the string is related to the length of the line segment in the X_1 direction,

$$\lambda_{CHL} = \left(\frac{R_1}{\ell_{11}}\right)^{\frac{3}{2}}.$$

In the weak coupling limit we therefore have to send $R_1 \to 0$. Let us see what this means in terms of the gauge theory. The torus of the gauge theory is the T-dual of the X_+X_- torus, which had the metric (2.2). T-dualising means inverting the metric and multiplying by $\alpha'^2 = \ell_{11}^6/R_{11}^2$, which yields

$$\hat{G} = \ell_{11}^6/R_{11}^2 \begin{pmatrix} R_2^{-2} + R_1^{-2} & R_2^{-2} - R_1^{-2} \\ R_2^{-2} - R_1^{-2} & R_2^{-2} + R_1^{-2} \end{pmatrix}. \tag{2.8}$$

So we see that in the weak coupling limit the radii of the torus remain equal, and go to infinity, but the angle between the two periodic coordinates goes to $-\pi$.

The fundamental domain (of the torus) therefore degenerates from a two-dimensional diamond to a one-dimensional line, which will become the string (figure 2). At the same time its surface, $\sqrt{\det \hat{G}}$ becomes infinite. In this IR limit the (dimensionful) gauge coupling constant also goes to infinity.

The fields living on the line are first of all the untwisted ones. The surviving ones are those that are independent of the coordinate transverse to the diagonal. We have eight scalars and eight right moving fermions, all in the symmetric representation of $O(N)$, and furthermore the gauge fields and adjoint spinors

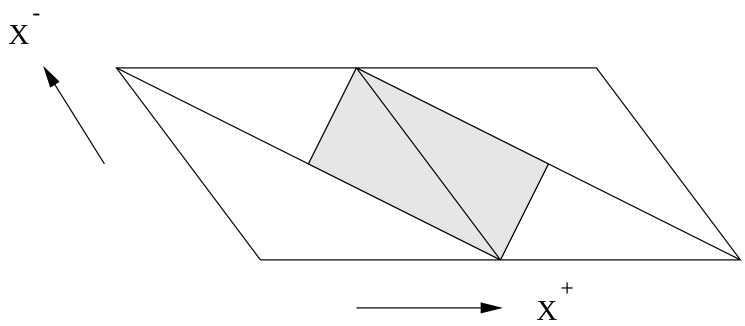

Figure 2: *In the weak string coupling limit, the strip collapses to a line*

which are left movers. Then there are the left moving twisted fermions in the fundamental of $O(N)$. There are sixteen of these, but their periods are twice those of the untwisted fields.

In the infrared limit the commutator of the X-fields is required to vanish, so the coordinate fields can be simultaneously diagonalised. At the same time the gauge multiplet decouples. The eigenvalues of the X's may be permuted by a Weyl reflection upon circling the string; one so obtains the long strings as twisted sectors. The left moving fundamental spinors are also exchanged by this Weyl reflection upon completing one cycle (which has double the length of a right moving cycle). Furthermore, the element -1 of $O(N)$ acts non-trivially on the fundamental spinors; we will have to project on the subspace of states invariant under this element. It is then clear how to obtain the CHL spectrum: in the sector with long strings of length n, the left moving bosons have moding $\frac{1}{n}$. Since the fermions' periods are twice those of the bosons, they have half the normal moding: in the anti-periodic sector (A) they have modes $\frac{1}{4n} + \frac{m}{2n}$, while in the P-sector their modes are $\frac{m}{2n}$. The left moving vacuum energy is $-\frac{1}{2n}$ in the A-sector, and 0 in the P-sector. States of zero mass therefore have $SO(16)$ adjoint quantum numbers in the A-sector, while in the P-sector they group together in a spinor and an anti-spinor of $SO(16)$. One of the latter two is projected out by the -1 element of $O(N)$, so that we are left with the adjoint representation of E_8, as expected for the CHL-string.

3. M-theory on a Klein bottle

For the construction of the matrix model compactified on a Klein bottle we start from IIA theory compactified on a two-torus, with radii R_1 and R_2, and then divide out the symmetry $\Omega\mathcal{I}_1\mathcal{S}_2$ [5, 6]. When we lift this to eleven dimensions, the operation $\Omega\mathcal{I}_1$ is the inversion of the eleventh coordinate. The topology of the M-theory compactification manifold is then indeed that of a Klein bottle. In terms

of the type IIA theory, the inversion of the eleventh dimension is interpreted as the operation $(-1)^{F_L}$, with F_L the left moving space-time fermion number. This can be checked by comparing the action of both operations on the massless fields: they multiply all RR-fields by -1. In the limit of small eleventh dimension we therefore expect to recover a weakly coupled type IIA string, modded out by the symmetry $\mathcal{S}_2(-1)^{F_L}$.

Let us first again review the orientifold model. In the closed string sector, the shift $\mathcal{S}_2$ acts as $(-1)^{n_2}$, with n_2 the momentum number along the second circle. If we are interested in the massless spectrum, we therefore have to find those states invariant under $\Omega\mathcal{I}_1$. This is easily done by performing a T-duality along the first circle. Then $\Omega\mathcal{I}_1$ is converted to Ω, and we obtain, in the massless sector, the closed string spectrum of the type I string. There are no closed string twisted sectors, since the only symmetry we divide out contains Ω. Then there could be open strings, but upon calculating the (world sheet) Klein Bottle contribution to the RR-tadpoles, one finds that this vanishes. The reason is that the momentum factor in the amplitude is of the form $\sum(-1)^n e^{-\frac{t\alpha' n^2}{R_2^2}}$. Poisson resummation shows that this vanishes in the $t \to 0$ limit. In conclusion, the theory is consistent without D-branes. This is consistent with the observation that the symmetry we divided out has no fixed points.

We now want to introduce zero-branes in this set-up. Their dynamics will be described by a certain $2+1$-dimensional gauge theory, which in the limit $N \to \infty$ should capture the dynamics of the whole theory. The usual strategy to find the appropriate gauge theory is to use T-duality, as we did in the previous section. There we could circumvent the problem of T-dualising the half shift $\mathcal{S}_2$ by adopting suitable new coordinates. In the present case however, we found no such obvious mechanism to T-dualise.

Instead, we will construct the corresponding two-dimensional gauge theory following the original strategy of [10, 11]. In the matrices X^μ for N zero-branes on a compact space we include entries for the images of the zero-branes under translation over the circles (and strings wrapping the circles), so that the X^μ are infinite-dimensional. The invariance under the translations then poses restrictions on the various sub-matrices. In the case of zero-branes on a torus, with no further symmetries divided out, one finds that in the compact directions the X^μ-matrices have the structure of a covariant derivative, acting on a field in the fundamental representation of $U(N)$. The various $N \times N$ blocks in the infinite matrices then correspond to the Fourier components of the gauge fields.

In the case at hand we also want to divide out the symmetry $\Omega\mathcal{I}_1\mathcal{S}_2$. The shift $\mathcal{S}_2$ has the effect of adding one to the indices denoting the image along the

two-direction, plus increasing the diagonal entry of X_2 by πR_2. Ω transposes the matrices, while $\mathcal{I}_1$ multiplies X_1, and the indices denoting the image along the one-direction, by -1.

We demand the matrices to be invariant under this transformation, which places restrictions on the various blocks. It turns out that we can again represent them as covariant derivatives, acting on two fields in the fundamental representation, one periodic along the x_2 direction with radius $\frac{1}{R_2}$, and the other antiperiodic:

$$\Phi = \left(\begin{array}{c} \phi^+(n_1,n_2)e^{2\pi i(n_1x_1R_1+n_2x_2R_2)} \\ \phi^-(n_1,n_2)e^{2\pi i(n_1x_1R_1+(n_2+\frac{1}{2})x_2R_2)} \end{array} \right).$$

The $X_{1,2}$ can be identified as the Fourier decomposition of the covariant derivatives

$$X_1 = -i\partial_1 + \left(\begin{array}{cc} A_1(x_1,x_2) & B_1(x_1,x_2) \\ B_1^\dagger(x_1,x_2) & -A_1^T(x_1,-x_2) \end{array} \right), \quad X_2 = -i\partial_2 + \left(\begin{array}{cc} A_2(x_1,x_2) & B_2(x_1,x_2) \\ B_2^\dagger(x_1,x_2) & A_2^T(x_1,-x_2) \end{array} \right), \tag{3.1}$$

with B_μ *antiperiodic* in the x_2 direction, and satisfying $B_\mu(x_1,x_2) = \pm B_\mu^T(x_1,-x_2)$, with the $-$ sign for $\mu = 1$, and the $+$ for $\mu = 2$. The other, non-compact, directions have the same structure as X_2, but lack of course the derivative.

In this representation, the translations along the torus are represented by the unitary transformations

$$U_1 = \left(\begin{array}{cc} e^{2\pi i x_1 R_1} & 0 \\ 0 & e^{2\pi i x_1 R_1} \end{array} \right), \quad U_2 = \left(\begin{array}{cc} e^{2\pi i x_2 R_2} & 0 \\ 0 & e^{2\pi i x_2 R_2} \end{array} \right),$$

while the operation $\mathcal{S}_2$ is given by

$$U_{\mathcal{S}_2} = \left(\begin{array}{cc} 0 & e^{\pi i x_2 R_2} \\ e^{\pi i x_2 R_2} & 0 \end{array} \right),$$

which duly squares to U_2.

The fermions behave similarly. We have 16 fermionic coordinates (matrices) S_α. The symmetries are the same, except that $\mathcal{I}_1$ acts as the ten-dimensional gamma matrix γ_1 on the ten-dimensional (chiral) spinor index. If we then choose a basis in which γ_1 is diagonal, we find that half of the fermions have the structure of X_1 (the gauginos) and half of them are in the other representation (matter).

The gauge theory describing M-theory on a Klein Bottle is quite strange. Whereas in the case of the Möbius strip we found a gauge theory whose base manifold was again a Möbius strip, in the Klein Bottle case this is definitely not so. Rather, we have gauge fields having some part (the A-field) living on a torus, without any further restriction beside being hermitian. The off-diagonal blocks, $B, B^\dagger$, the antiperiodic modes of the gauge field (or the odd modes when going

to a circle of double the radius), only have independent components on half of the torus; effectively they live on a cylinder, and are forced to be symmetric on one boundary and anti-symmetric on the other.

We now still want to identify the limit of the theory in which it describes a weakly coupled type IIA string. This limit corresponds to $R_1 \to 0$. In the gauge theory we see that this implies that x_1's period goes to infinity. So again, as expected, we find that the dimensionless gauge coupling diverges, and that only the zero modes in the x_2 direction survive. This means that we effectively obtain a two-dimensional gauge theory, which we interpret as the world sheet theory of a string. Since the B fields are anti-periodic in the x_2 direction, they do not have zero modes, so their masses go to infinity and they can be ignored. The massless fields that remain on the world sheet are therefore the regular unitary components A. The world sheet theory is therefore that of the type IIA string, as we expected.

Then we have to show that this theory indeed is invariant under the extra symmetry. First of all it is clearly invariant under the large gauge transformation U_2, identifying X_2 with $X_2 + 2\pi R_2$, so that the theory indeed lives on a circle. To see that the other symmetry indeed reduces to $\mathcal{S}_2(-1)^{F_L}$, note that in the strong coupling gauge theory we are considering, the vanishing of the potential implies that all the fields can be simultaneously diagonalised. Therefore the matter fields are invariant under $U_{\mathcal{S}_2}$, sending X_2 to $X_2 + \pi R_2$. The gauge field and the gauginos, on the other hand, get a -1, so to have a full symmetry we need to add to $\mathcal{S}_2$ a transformation multiplying these by an extra -1. In the strong coupling limit we are considering, the gauge field drops out of the action, so basically we have to check that the fermions with γ_1 equal to -1 are precisely the fermions of definite chirality in the dimensionally reduced gauge theory.

This can be easily verified by writing the zero-brane lagrangian and inserting the solutions for the X_μ and S_α. In a basis for the ten-dimensional gamma-matrices with

$$\Gamma_0 = \mathbf{1} \otimes i\sigma_2, \quad \Gamma_i = \gamma_i \otimes \sigma_1,$$

with γ_i a set of nine-dimensional gamma-matrices, we have that the ten-dimensional chiral spinors are of the form $(S_\alpha, 0)$. The interaction term in the matrix model lagrangian is then of the form

$$\mathrm{Tr}\left(-S^T \gamma^i [X_i, S]\right). \tag{3.2}$$

Inserting the representations we found above, the $1+1$-dimensional fermion derivative terms reduce to

$$\int dx_1 \mathrm{Tr}\left(-iS^T(\partial_0 - \gamma^1 \partial_1)S\right), \tag{3.3}$$

so that indeed γ^1 determines the two-dimensional chirality of the spinor. In the weak coupling limit the symmetry therefore can be correctly identified as $\mathcal{S}_2(-1)^{F_L}$.

4. Conclusion

We have derived the gauge theory models describing matrix theory on two non-orientable surfaces, the Möbius strip and the Klein bottle. The Möbius gauge theory lives on a Möbius strip itself. In the bulk we have a $U(N)$ $2+1$-dimensional gauge theory. On the boundary the symmetry is reduced to $O(N)$, with a matter multiplet in the symmetric representation. Anomaly cancellation requires extra twisted chiral fields on the boundary. In the limit of a small line segment the chiral fermions represent the level two E_8 current algebra living on the CHL string.

For the Klein bottle compactification, the T-dualisation necessary for finding the gauge theory was less straightforward; we had to find a covariant derivative representation for the zero-brane coordinate matrices "by hand". The base manifold of the gauge theory turns out to be a hybrid of a torus for the periodic modes of the fields and a cylinder for the anti-periodic modes in the x_2-direction. Alternatively one might describe the theory as living on a cylinder, with two gauge fields that are identified on the boundaries, and another field that on one boundary is symmetric, on the other anti-symmetric. In the limit of weak type IIA string coupling, the anti-periodic modes do not survive the dimensional reduction, and we are left with a type IIA matrix string invariant under the extra symmetry $\mathcal{S}_2(-1)^{F_L}$.

Acknowledgements

The author is indebted to Erik Verlinde for advice and discussions, and to FOM for financial support.

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